# ON THE INDIVIDUAL ERGODIC THEOREM IN F-QUANTUM SPACES

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Inspiring by the Piasecki measure theory [1] and the quantum logic theory [2], there was proposed in [3], [4] a new model of statistical quantum mechanics, so-called F-quantum spaces. In [5] and [6] there were achieved some basic results concerning probability distributions on F-quantum spaces. Using these results we shall prove in this article the individual ergodic theorem for F-quantum spaces. Recall that some versions of the individual ergodic theorem in quantum logics has been proved in [7] – [9].

## Definitions and notation

We shall use a usual notation of the fuzzy set theory. So a subset $M$ of a non-empty set $X$ is a function $U: X \to [0, 1]$. For two fuzzy subset $u, v$ of $X$ we put $u \vee v = \max(u, v)$, and for a sequence $(u_n)_n$ *of* fuzzy subset of $X$ we define $\bigvee u_n = \sup u_n$. The complement of fuzzy set $u$ is the function $1 - u$.

By $B(R^1)$ or $B(R^2)$ we shall denote the $\sigma$-algebra of Borel subsetof $R^1$ or $R^2$, respectively.

F-quantum space is a couple $(X, M)$, where $X$ is a non-empty set $M \subset [0, 1]^X$ such that the following conditions are satisfied: a) $1_X \in M$; b) $f \in M \Rightarrow 1 - f \in M$; c) $f_n \in M$ $(n = 1,2,\ldots,) \Rightarrow \sup f_n \in M$; d) If $g(x) = \frac{1}{2}$ for every $x \in X$, then $g \notin M$.

F-state on an F-quantum space $(X, M)$ is a mapping $m: M \to [0, 1]$ such that a) $m(f \vee (1-f)) = 1$ for every $f \in M$; b) If $f_i \in M$ $(i = 1,2,\ldots,)$ and $f_i \leqq 1 - f_j$ $(i \neq j)$, then $m(\bigvee f_i) = \sum m(f_i)$. F-observable on an F-quantum space $(X, M)$ is a mapping $x: B(R^1) \to M$ satisfying the following properties: a) $x(A') = 1 - x(A)$ for every $A \in B(R^1)$; b) If $A_n \in B(R^1)$, then $x(\bigcup A_n) = \bigvee x(A_n)$.

If $m: M \to [0, 1]$ is an F-state and $x: B(R^1) \to M$ is an F-observable, then we define $m_x: B(R^1) \to [0, 1]$ by the formula $m_x(E) = m(x(E))$. Evidently $m$ is a probability measure.

Recall that similarly as in the quantum logic case these definitions are generalization of some classical ones. Namely, if $(\Omega, S, P)$ is a probability space and $\xi:\Omega\to R$ is a random variable, then the mapping $x:B(R^1)\to S$ assigning to every $E\in B(R^1)$ the set $\xi^{-1}(E)$ is a $\sigma$-homomorphism. So substituting $S$ by $M$ we ohtain the definition of an F-observable.

## A representation theorem

In the proof of some variants of the individual ergodic theorem in [7] – [9] a representation lemma played an important role. A corresponding modification for F-quantum spaces has been proved in [10].

THEOREM 1. Let $y, z:B(R^1)\to M$ be two F-observables, $z(B(R^1))\subset y(B(R^1))$. Then there is a Borel measurable mapping $T:R^1\to R^1$ such that $z(E)=y(T^{-1}(E))$ for every $E\in B(R^1)$.

## The calculus for observables

Two observables $x, y$ have a joint observable, if there is a $\sigma$-homomorphism $\varkappa:B(R^2)\to M$ such that $\varkappa(A\times B)=x(A)\wedge y(B)$.

The following theorem has been proved in [6].

THEOREM 2. Two observables $x$, y have a joint observable if and only if $x(\emptyset)=y(\emptyset)$.

If $x, y$ have a joint observable, then we can define its sum, difference etc. Put. e.g. $g(u,v)=u+v$, $h(u,v)=u-v$ $(u,v\in R^1)$. Then $g,h:R^2\to R^1$ are Borel measurable maps and we define $x+y=\varkappa\circ g^{-1}$, $x-y=\varkappa\circ h^{-1}$. The motivation is the following. If $\xi$, $\eta$ are two random variables define on a probability space $(\Omega, S, P)$, then

$$\xi+\eta=g\circ T,$$

where

$$T=(\xi,\eta):\Omega\to R^2.$$

Then

$$(\xi+\eta)^{-1}(E)=T^{-1}(g^{-1}(E))$$

and $\varkappa=T^{-1}:B(R^2)\to S$ is the corresponding $\sigma$-homomorphism.

Another kind of operation is the multiple of an F-observable. If $x:B(R^1)\to M$ is an F-observable and $c\in R^1$, $c\neq 0$, then $cx:B(R^1)\to M$ is defined by the formula $cx(E)=x\left(\frac{1}{c}E\right)$, where $\frac{1}{c}E=\left\{\frac{1}{c}t;\ t\in E\right\}$.

A special situation is described in the following proposition.

PROPOSITION 1. Let $u, v, y:B(R^1)\to M$ be F-observables, $f,g:R^1\to R^1$ be Borel measurable maps such that $u=y\circ f^{-1}$, $v=y\circ g^{-1}$. Then $u+v=y\circ(f+g)^{-1}$.

Proof. The observables $u, v$ have a joint observable, since $u(\emptyset)=$ $=y(f^{-1}(\emptyset))=y(\emptyset)$ and $v(\emptyset)=y(g^{-1}(\emptyset))=y(\emptyset)$. Put $h=(f,g): R^1\to R^2$. By Theorem 2 there is a $\sigma$-homomorphism $\varkappa: B(R^2)\to M$ such that

$$\varkappa(A\times B)=u(A)\wedge v(B)$$

for every $A, B\in B(R^1)$. So by the assumption

$$\varkappa(A\times B)=y(f^{-1}(A)\cap g^{-1}(B))=y(h^{-1}(A\times B)).$$

This implies that

$$\varkappa(E)=y(h^{-1}(E))$$

for every $E\in B(R^2)$. Put $\varphi(x, y)=x+y$. Then by the definition

$$u+v(E)=\varkappa(\varphi^{-1}(E)=y(h^{-1}(\varphi^{-1}(E)))=$$
$$=y((\varphi_0 h)^{-1}(E))=y((f+g)^{-1}(E))$$

and Proposition 1 is proved.

## Convergence

Let $(x_n)_n$ be a sequence of F-observables and $x$ be such an F-observable, that $x_n(\phi)=x(\phi)$ (so that $x_n, x$ has a joint observable) $(n=1, 2, \ldots.)$, $m: M\to$ $[0, 1]$ be an F-state. We say that $x_n\to x[m]$, if

$$m\Big(\bigwedge_{k=1}^{\infty}\bigvee_{n=k}^{\infty}(x_n-x)([-\varepsilon, \varepsilon]')\Big)=0$$

for every $\varepsilon>0$.

It is not difficult to see that a sequence $(\xi_n)_n$ of random variables converges to a random variable $\xi$ almost everywhere if and only if

$$P\Big(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}(\xi_n-\xi)^{-1}([-\varepsilon, \varepsilon]')\Big)=0$$

for every $\varepsilon>0$. The following proposition is a simple aplication of this remarque.

Proposition 2. Let $(x_n)_n$ be a sequence of observables such that $x_n=y\circ f_n^{-1}$, $x=y\circ f^{-1}$, where $f_n, f: R^1\to R^1$ are Borel measurable functions. Then $x_n\to x\,[m]$ if $f_n\to f$ $m_y$-almost everywhere, where $m_y(E)=m(y(E))$.

Proof. Since $x_n(\emptyset)=y(\emptyset)=x(\emptyset)$, the difference $x_n-x$ is well defined. Then by Proposition 1

$$m\Big(\bigwedge_{k=1}^{\infty}\bigvee_{n=k}^{\infty}(x_n-x)([-\varepsilon, \varepsilon]')\Big)=m(\wedge\vee y\circ(f_n-f)^{-1}([-\varepsilon, \varepsilon]'))=$$
$$=m(y(\cap\cup(f_n-f)^{-1}([-\varepsilon, \varepsilon]')))=m_y\Big(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}(f_n-f)^{-1}([-\varepsilon, \varepsilon]')\Big).$$

## Ergodic theorem

For to formulate the ergodic theorem we need the following definitions. An F-observable $x$ is said to be integrable, if there exists the integral

$$m(x)=\int_{R^1} t\, dm_x(t),$$

where $m_x$: $B(R^1)\to[0, 1]$, $m_x(E)=m(x(E))$.

Let $y$: $B(R^1)\to M$ be an F-observable, $\tau$ is called y-measurable if $\tau(y(B(R^1)))\subset y(B(R^1))$. If $m(\tau(f))=m(f)$ for every $f\in M$, then $\tau$ is called an invariant $\sigma$-homomorphism (in a state $m$).

THEOREM 3. Let y be an F-observable, $\tau$: $M\to M$ a y-measurable, invariant (in a state $m$) $\sigma$-homomorphism. Let $x$ be an integrable F-observable such that $x(B(R^1))\subset y(B(R^1))$, $\tau(x(\emptyset))=x(\emptyset)$. Then there exists and F-observable $x^*$ such that

1. $x^*$ is integrable, $m(x^*)=m(x)$.
2. $\tau_\circ\, x^*=x^*$.
3. $\frac{1}{n}\sum_{i=0}^{n-1}\tau^i x\to x^*\,[m]$.

Proof. If we put $z=x$ in Theorem 1, then we obtain by the inclusion $x(B(R^1))\subset y(B(R^1))$ that there is a Borel measurable mapping $f$: $R^1\to R^1$ such that

$$x(E)=y(f^{-1}(E)) \tag{1}$$

for every $E\in B(R^1)$. Putting $z=y$ we obtain by the $y$-measurability of $\tau$ that there is a Borel measurable mapping $T$: $R^1\to R^1$ such that

$$\tau(y(E))=y(T^{-1}(E)) \tag{2}$$

for every $E\in B(R^1)$. By (1) and (2) we obtain

$$\tau(x(E))=\tau(y(f^{-1})(E)))=y(T^{-1}(f^{-1}(E)))=y((f_\circ T)^{-1}(E))$$

and similarly

$$\tau^i(x(E))=y((f_\circ T^i)^{-1}(E)). \tag{3}$$

Consider now the probability space $(R^1, B(R^1), m_y)$ and the random variable $f$: $R^1\to R^1$. By the integral transformation formula

$$\int_A g\, d\mu\circ f^{-1}=\int_{f^{-1}(A)} g\circ f\, d\mu$$

and (1) we obtain

$$m(x)=\int_{R^1} t\, dm_x(t)=\int_{R^1} t\, dm_y\circ f^{-1}(t)=\int_{R^1} f\, dm_y. \tag{4}$$

It follows that $f$ is $m_y$-integrable. By the individual ergodic theorem (e.g. [11])

there is an integrable random variable $f^*$ such that

$$\int_{R^1} f^* \, dm_y = \int_{R^1} f \, dm_y, \tag{5}$$

$$f^* \circ T f^*, \tag{6}$$

$$f^*(t) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i(t)) \qquad m_y - a.e. \tag{7}$$

Put $x^* = y \circ (f^*)^{-1}$. Then by (4) and (5)

$$m(x) = \int_{R^1} f \, dm_y = \int_{R^1} f^* \, dm_y = \int_{R^1} t \, dm_y (f^*)^{-1} = \int_{R^1} t \, dm_{x^*} = m(x^*).$$

Further by (2) and (6)

$$\tau(x^*(E)) = \tau\big(y((f^*)^{-1}(E))\big) = y(T^{-1}((f^*)^{-1}(E))) =$$
$$= y\big((f^* \circ T)^{-1}(E)\big) = y\big((f^*)^{-1}(E)\big) = x^*(E).$$

We have to prove an analogue of (7). First by (3)

$$\tau^i(x(\emptyset)) = y(\emptyset), \qquad i = 0, 1, \ldots,$$

hence by Theorem 2 we can produce

$$x_n = \frac{1}{n} \sum_{i=0}^{n-1} \tau^i \circ x.$$

If we put

$$S_n = \frac{1}{n} \sum_{i=0}^{n-1} f \circ T^i,$$

then by (3) and Proposition 1

$$x_n = y \circ S_n^{-1}. \tag{8}$$

Now (8), (7) and Proposition 2 give

$$x_n \to y \circ (f^*)^{-1} = x^* [m].$$

## References


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