A NOTE ON PIASECKI'S P-MEASURE

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Let σ be a family of fuzzy subsets of a universum Ω (i.e. mappings $\Omega \to [0, 1)$ containing 0_Ω, 1_Ω, not containing $\{\frac{1}{2}\}_\Omega$ and closed under countable union and complement. Such families are called by Piasecki [1] soft fuzzy σ-algebras.

DEFINITION [1]. By W-empty fuzzy subset of Ω we will mean a mapping $\mu : \Omega \to [0, 1]$ such that $\mu \leqslant \mu'$ ($\mu' = 1 - \mu$).

DEFINITION [1]. By W-universum in Ω we will mean a mapping $\mu : \Omega \to [0, 1]$ such that $\mu \geqslant \mu'$.

DEFINITION [1]. Two fuzzy subsets μ and ν of Ω are called w-separated sets if $\mu \leqslant \nu'$.

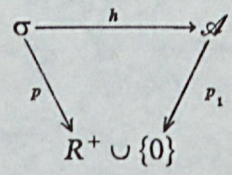
DEFINITION [1]. Let σ be a soft fuzzy σ-algebra. A mapping $p : \sigma \to \to R^+ \cup \{0\}$ is called a P-measure on σ if

1) $p(\mu) = 1$ for any W-universum μ,

2) $p(\bigcup_n \mu_n) = \sum_n p(\mu_n)$ for each countable family of pairwise W-separated sets from σ.

The representation theorem for the P-measure runs as follows.

THEOREM. For any soft fuzzy σ-algebra σ and any P-measure p on σ there exists a σ-Boolean algebra $\mathscr{A}$, a σ-De Morgan algebras homomorphism $h : \sigma \to \mathscr{A}$ and a usual probability measure $p_1 : \mathscr{A} \to R^+ \cup \{0\}$ such that the following diagram commutes



(1)

Proof. We will give only a brief sketch of the proof. Let W_0 be the class of all W-empty subsets from σ and let $\sim$ denotes the following relation in σ:

$$\mu \sim \nu \Leftrightarrow (\mu \wedge \nu') \vee (\nu \wedge \mu') \in W_0 .$$

The relation $\sim$ is a tolerance on σ (reflexive and symmetric) and may be not transitive. Let $\approx$ be the transitive closure of $\sim$. From the results of Piasecki

[1] it can be checked (this part of the proof is omitted here) that $\approx$ is consistent with the operations of taking countable sum (intersection) and complement. We have also $\mu \vee \mu' \approx 1$, $\mu \wedge \mu' \approx 0$ and

$$\mu \approx \nu \Rightarrow p(\mu) = p(\nu).$$

Hence $\mathscr{A} = \sigma/\approx$ is a σ-Boolean algebra and, if we define $h(\mu) = [\mu]$, $p_1([\mu]) = p(\mu)$, then h is a σ-De Morgan algebras homomrphism, p_1 is a usual probability measure on $\mathscr{A}$ (i.e. $p_1(1) = 1$ and $p_1(\bigcup_n \mu_n) = \sum p_1(\mu_n)$ for any countable family μ_n of pairwise disjoint elements from $\mathscr{A}$) and the diagram (1) commutes. This ends the proof.

By the above theorem we can derive all the properties of P-measure established by Piasecki [1] in a very easy way. For example the continuity from below of p (Theorem 3.4. in [1]) is a consequence of continuity from below of p_1 and commutation of h with countable unions, namely

$$p(\bigcup_n \mu_n) = p_1 h(\bigcup_n \mu_n) = p_1[\bigcup_n h(\mu_n)] = \lim_{n \to \infty} p_1[h(\mu_n)] = \lim_{n \to \infty} p(\mu_n).$$

Other results of [1] can be obtained analogously with the help of (1).

Reference

[1] K. Piasecki, Probability of fuzzy events defined as denumerable additivity measure, Fuzzy Sets and Systems 17 (1985) 271-284.