ON RELATIONSHIP BETWEEN FUZZY AND RANDOM SETS.

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1. The basic ideas of this paper arise from relationship between fuzzy and random sets. The different operations of fuzzy sets is defined and the two construction of random set equivalent to a fuzzy set under one-point coverages is presented. The problem of selection of random set belonging to the same equivalent class is determined for two selection criteria: maximum probability-possibility consistency and maximum entropy. This problem is connected with generating single value of the fuzzy variable. The relation between operation of fuzzy and random sets is presented.

2. Let X be an universal space and, A, B be the fuzzy subsets of this space. We define two operations of fuzzy sets, denoted by symbols $\perp$ and T, in a such way:

$$(A \perp B)(x) = s(A(x),\ B(x)),$$

$$(A \mathsf{T} B)(x) = t(A(x),\ B(x)),$$

where s: $[0,\ 1]\times[0,\ 1]\to[0,\ 1]$ is a nondecreasing in each argument, commutative, associative function, which has 1 as unit, and $t(x,\ y)=1-s(1-x,\ 1-y)$. The function s is called t-conorm and t is called t-norm. [6].

The operations $\perp$ and T are the generalization of union and intersection of usual sects. For instance, in the case when $s_1(x,\ y)=x\vee y$, then $t_1(x,\ y)=x\wedge y$ and $\perp$,T are the standard union $\cup$ and $\cap$ intersection defined by L. A. Zadeh [8]. If $s_2(x,\ y)=x+y-xy$ we obtain $t_2(x,\ y)=xy$ and sum and product [5] of fuzzy set denoted by $\hat{+}$ and $\cdot$ respectively.

Let us assume that the space X is finite. The function s is assotiative and we can define algebraic sum of fuzzy sets:

$$(A+B)(z)=s\big(t(A(x_1),\ B(y_1)),\ldots,\ t(A(x_m),\ B(y_m))\big)$$

for all x_i, y_i such that $x_i+y_i=z$.

In the case when $s=s_1$ we have a standard algebraic sum:

$$(A\oplus B)(z)=\sup\{A(x)\wedge B(y):x+y=z\},$$

and when $s=s_2$ we have another sum denoted by symbol $\boxplus$. These are the generalization of complex (Minkowsky) sum for usual subsets.

3. Let Y be a measurable map from probability space $(\Omega, \mathfrak{M}, \Pr)$ to the class $\mathscr{P}(X)$ of all subsets of finite space $X=\{x_1, \ldots, x_n\}$. This map we call a random subset of space X. We denote by symbol $L(X)$ the class of all random sets of space X.

The probability measure Pr induces in the usaul way the four set functions on class $\mathscr{P}(X)$: [7]

$$m(A)=Pr\{Y=A\} \text{ – basic probability assignment,}$$
$$\bar{P}(A)=Pr\{Y\cap A=\emptyset\} \text{ – upper measure,}$$
$$\underline{P}(A)=Pr\{Y\subset A\} \text{ – lower measure,}$$
$$\bar{Q}(A)=Pr\{A\subset Y\} \text{ – cover measure,}$$

where $A\in\mathscr{P}(X)$. Obviously they are not additive set functions i.e. probability measures.

The random sets are independent iff σ-fields generated by them are independent. The subset A for which $m(A)>0$ is called a focal element.

The space X is finite, the class $\mathscr{P}(X)$ is isomorfic with Boolean algebra 2^n and we can identify a random set Y with random variable taking value in this algebra. In this case $\underline{P}(A)$ is a measure of ideal generated by set A, and $\bar{Q}(A)$ is a measure of such filter [4].

4. When we have a random set Y we can generate a fuzzy subset $f(Y)$ of space X. The grade of membership of elements x to the fuzzy set $f(Y)$ is equal to the probability of event that x belongs to this random set: [3]

$$f(Y)(x)=Pr\{x\in Y\}=\bar{Q}(\{x\}). \tag{1}$$

In this manner we obtain the map f from the class of all random sets $L(X)$ to the class $F(X)$ of all fuzzy subsets of space X. It is easy to see, that the map f is not one-to-one, because formula (1) determines probability measure on the main filter generated by one-element sets only. The map f devides $L(X)$ on the equivalence classes containing the random sets generated the same fuzzy set. This random sets have the same one-point cover function $\bar{Q}(\{x\})$.

Example 1.

Let G be a fuzzy subset of X and U be a random variable uniformly distributed on the interval $[0, 1]$ and H_G be a multivalued mapping from $[0, 1]$ to $\mathscr{P}(X)$, defined by:

$$H_G(r)=G^{-1}([r, 1])=G_r,$$

where G_r is r-cut of fuzzy set G. The measurable multivalued map $S(G)=H_G\circ U$ is a random set satisfying an equality:

$$Pr\{x\in S(G)\}=G(x).$$

In the other words, the random set $S(G)$ induces the fuzzy set G.

In the case when space X is finite, the focal elements of the random set $S(G)$ are the r-cuts of fuzzy set G. They are nested, i.e. they can be arranged in order: $A_1 \subset A_2 \subset \ldots \subset A_m$, where A_i is a r_i-cut of fuzzy set G. Such random set is called a consonant. The basic probability assignment of this random set is determined by membership function of fuzzy set G in the following manner:

$$m(A_i) = r_i - r_{i-1}.$$

Example 2.

Let $\{V_x : x \in X\}$ be the family of independent random variables taking value in the two-element set $\{0, 1\}$ satisfying the following conditions:

$$Pr\{V_x = 1\} = G(x), \qquad Pr\{V_x = 0\} = 1 - G(x),$$

for any $x \in X$, and $Ch(X)$ be a space of characteristic function on X. The family V_x induces on $Ch(X)$ a product measure v_G. We identify this measure with distribution function of random set $T(G)$.

In the case when space X is finite, the random set $T(G)$ has the basic probability assignment equal:

$$m(A) = \prod_{x \in A} G(x) \prod_{x \notin A} (1 - G(x)).$$

These two random sets belong to the same equivalence class.

5. Let us assume that we have the following evidence: [2]

$$g\colon V \text{ is } G;$$

where V is a variable taking value on a realization set X and G is a normal, convex and bounded fuzzy subset of X. Such an evidence induced the possibility distribution of the variable V according to the possibility assignment equation:

$$Poss\{V = x\} = G(x).$$

A single value of this fuzzy variable can be generated in the following manner: [2]

a) generate a value of random set induced by fuzzy set G,

b) generate a single value of the uniform random variable over the set determined in step a).

This procedure is equivalent to a random generation of the single value of the variable V according to the probability distribution:

$$Pr\{V = x\} = \sum_{x \in A \in \mathcal{P}(X)} m(A)/\mathrm{card}(A), \tag{2}$$

where m is a basic probability assignment of random set generated by fuzzy set G.

6. Let G be a fuzzy subset of finite space X. Denote by $L(G)$ a class of all random sets generated by fuzzy set G. We determine the problem of the selection of random set belonging to $L(G)$ in case of two selection criteria:

a) criterion of maximum consistence between the probability distribution (2) and the possibility distribution induced by evidence V is G,

b) criterion of maximum entropy of the distribution of random set.

THEOREM 1. [1] The random set $S(G)$ has the maximum coeficent of the probability-possibility consistence $a(m)=\sum_{x\in X} Pr\{V=x\}\,G(x)$, where m is the basic probability assignment of random set $S(G)$, and the random set $T(G)$ has the maximum entropy among all sets belonging to $L(G)$.

Proof of this theorem avails of the Boolean representation of random sets (see [4]).

7. Now we consider the problem of the relationship between operation of fuzzy and random sets. Let A, B be fuzzy sets, $\cup$, $\cap$, $\hat{+}$, . be two generalization of union and intersection of fuzzy sets and $\oplus$, $\boxplus$ be generalization of algebraic sum defined in section 2. The following theorems are true:

THEOREM 2. If random sets $S(A)$ and $S(B)$ are generated by the same random variable then

a)
$$S(A\cup B)=S(A)\cup S(B), \qquad S(A\cap B)=S(A)\cap S(B),$$

b)
$$S(A\oplus B)=S(A)+S(B),$$

where $+$ is a complex sum of random sets.

THEOREM 3. If random sets $T(A)$ and $T(B)$ are independent then

a)
$$T(A\,\hat{+}\,B)=T(A)\cup T(B), \qquad T(A\cdot B)=T(A)\cap T(B),$$

b)
$$A\boxplus B=f(T(A)+T(B)).$$

It is easy to see that property b) in theorem 3 is weaker than analogical property in theorem 2. That property only guarentees that random sets $T(A\boxplus B)$ and $T(A)+T(B)$ belong to the same equivalence class.

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