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THE RELATION OF DOUBLING IN COMMON FIXED POINT THEORY

The concept of a 2-metric space[1], introduced by S. Gähler [1], provides a natural abstraction of the area function for Euclidean triangles.

Recently, Khan and Fisher [2] have introduced a necessary and sufficient condition which guarantees the existence of a common fixed point for a pair of continuous mappings in a 2-metric as well as in a metric space.

In this paper we will prove similar theorem in probabilistic 2-metric spaces and we will extend fixed point theorems of Rhoades [3], Singh-Tiwari-Gupta [6]. In line with [1] and [3], we will give some definitions.

A distance distribution function (d.d.f.) is a non-decreasing function $F: [0, +\infty] \to [0, 1]$, which is left-continuous on $(0, +\infty)$ and assumes the values $F(0)=0$ and $F(+\infty)=1$. The set of all d.d.f's denoted by Δ^+, is equipped with modified Levy metric d_L (see p. 45 of [5]). The metric space

[1] Let X be a non-empty set and d: $X \times X \times X \to [0, +\infty)$ a function satisfying the following axioms:

(i) $d(x, y, z)=0$ if at least two of x, y, z are equal,

(ii) for each two distinct points $x, y \in X$, there exists a point $z \in X$ such that $d(x, y, z) \neq 0$,

(iii) $d(x, y, z)=d(x, z, y)=d(z, y, x)$,

(iv) $d(x, y, z) \leqslant d(x, y, w)+d(z, w, z)+d(w, y, z)$, for all $x, y, z, w \in X$.

The function d is called a 2-metric on X, and (X, d) is called a 2-metric spaces.

$(\Delta^+, d+L)$ is compact, and hence complete, and Δ^+ is partially ordered by the usual order for real-valued functions.

Let u_a be the element of Δ^+ defined by

$$u_a = \begin{cases} 1_{(a, +\infty)}, & \text{for all } a \in [0, +\infty), \\ 1_{\{+\infty\}}, & \text{for } a = \{+\infty\}. \end{cases}$$

Definition 1. A probabilistic 2-metric space (briefly 2-PM-space) is an ordered pair $(X, \mathcal{F})$ where X is a non-empty set and the mapping $\mathcal{F}: x^3 \to \Delta^+$ is assumed to satisfy the following conditions:

(F0) $F_{xyz}(0) = 0$ for all x, y, z.

(F1) $F_{xyz} = \varepsilon_0$ if $x = y$ or $y = z$ or $z = x$.

(F2) $F_{xyz} = F_{xzy} = F_{yzx}$.

(F3) For two distinct points x, y in X there is a point $z \in X$ such that $F_{xyz} \neq \varepsilon_0$.

(F4) If $F_{xyw}(t) = 1$ and $F_{xwz}(l) = 1$ and $F_{wyz}(s) = 1$, then $F_{xyz}(t+l+s) = 1$, for all $t, s, l > 0$.

Definition 2. A probabilistic 2-Menger space (2-M-space) is an ordered triple $(X, \mathcal{F}, T)$, where $X \neq \emptyset$ if the function $\mathcal{F}: X^3 \to \Delta^+$ satisfies the conditon (F0-F3) and

(F4M) $$F_{xyz}(t+s+l) \geqslant T[F_{xyw}(t), F_{xwz}(s), F_{wyz}(l)], \quad t, s, l > 0,$$

where the function $F: [0, 1]^3 \to [0, 1]$ has the following properties [T is called a t_2-norm]:

(T1) $T(a, b, c) = T(b, c, a) = T(a, c, b)$,

(T2) $T(a, b, c) \leqslant T(a_1, b_1, c_1)$ if $a \leqslant a_1$, $b \leqslant b_1$ and $c \leqslant c_1$,

(T3) $T(a, 1, 1) = a$ for all $a \in [0, 1]$,

(T4) $T(a, b, T(c, d, e)) = T(T(a, b, c), d, c)$.

Remark 1. The function $T(\cdot, \cdot, 1): [0, 1] \times [0, 1] \to [0, 1]$ is a simple t-norm. On the other hand, if the function L is a t-norm, then the function T defined by

$$T(a, b, c) = L(a, L(b, c))$$

satisfies the axioms (T1-T4).

Remark 2. Every 2-metric space (X, d) may be regarded as an 2-PM space of a special kind. We only have to show that

$$F_{xyz}(t)=\varepsilon_0(t-d(x, y, z))$$

for every triple $x, y, z \in X$ is representative of a 2-metric space.

Definition 3. A sequence $\{x_n\}$ in 2-PM space $(X, \mathcal{F})$ is said to be convergent with limit x in X if $\lim_{n\to\infty} F_{x_n xa}(t)=1$ for all $a \in X$ and $t>0$.

A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if $\lim_{n, m\to\infty} F_{x_m x_n a}(t)=1$ for all a in X and $t>0$. The 2-PM space in which every Cauchy sequence in convergent is called complete.

LEMMA 1. *If T is continuous and $x_n \to x$ then*

$$\lim_{n\to\infty} \inf F_{x_n ab}(t)=F_{xab}(t)$$

for all $a, b \in X$ and $t>0$.

Corollary 1. If $F_{xyz}(\cdot)$ is continuous for all x, y, z in X, then

$$\lim_{n\to\infty} F_{x_n yz}(t)=F_{xyz}(t).$$

Definition 4. A 2-PM space F on a set X is said to be lower-semicontinuous on X if it is sequentially continuous in two of its three arguments and continuous, if F_{xyz} is continuous for $x, y, z \in X$.

It follows that if $\mathcal{F}$, is sequentially continuous in two of its three arguments it is sequentially continuous in all three arguments.

THEOREM 1. *Let S and R be continuous mappings of a complete 2-PM space $(X, \mathcal{F})$ into itself and let $\mathcal{F}$ be continuous (lower-semi continuous) and $\lim_{t\to\infty} F_{abc}(t)=1$ for all $a, b, c \in X$ and $t>0$. Then S and R have a common fixed point in X if and only if there exists a continuous mapping A of X into $SX \cap TX$, which commutes wich S and T and satisfies the inequality*

$$F_{Ax, Ay, a}(k) \geqslant \operatorname{Min}[F_{Ax, Sx, a}(t), F_{Sx, Ry, a}(t), F_{Ay, Ry, a}(t)],$$

for all x, y, a in X where $0<k<1$.

Indeed, A, S and R then have a unique common fixed point.

Proof. To prove the necessity of the condition, suppose that S and T have a common fixed point z. Let us define a mapping A of X into $SX \cap TT$ by

putting $Ax=z$, for all x in Y. Then A commutes with S and T and for any k with $0<k<1$

$$1=F_{Ax,Ay,a}(kt)\geqslant \mathrm{Min}[F_{Ax,Sx,a}(t),\ F_{Sx,Ty,a}(t),\ F_{Ay,Ty,a}(t)],$$

for all x, y, a in X and $t>0$.

For the sufficiency of the condition, let us suppose that there exists a mapping A with the properties stated in the theorem. Let x_0 be an arbitrary point in X. As Ax is contained in SX, we can choose a point x_1 in X such that $Sx_1=Ax_2$.

Since Ax is also contained in TX we can choose a point x_2 in X with $Tx_2=Ax_1$. In general, we can choose a point x_{2n+1} such that $Sx_{2n+1}=Ax_{2n}$ and $Tx_{2n+1}=Ax_{2n+1}$ for $n=0,\ 1,\ 2,\ \ldots$

$$\begin{aligned}F_{Ax_{2n+1},Ax_{2n},a}(kt)&\geqslant \mathrm{Min}[F_{Ax_{2n+1},Sx_{2n+1},a}(t),\ F_{Sx_{2n+1},Tx_{2n},a}(t),\ F_{Ax_{2n},Tx_{2n},a}(t)]=\\&=\mathrm{Min}[F_{Ax_{2n+1},Ax_{2n},a}(t),\ F_{Ax_{2n},Ax_{2n-1},a}(t),\ F_{Ax_{2n},Ax_{2n-1},a}(t)]\geqslant\\&\geqslant F_{Ax_{2n},Ax_{2n-1},a}(t),\end{aligned}$$

$$F_{Ax_{2n+1},Ax_{2n},a}(kt)\geqslant F_{Ax_{2n},Ax_{2n-1},a}(t),\qquad \text{when}\quad 0<k<1.$$

Similarly, $F_{Ax_{2n},Ax_{2n-1},a}(kt)\geqslant F_{Ax_{2n-1},Ax_{2n-2},a}(t)$ and so

$$F_{Ax_{2n+1},Ax_n,a}(t)\geqslant F_{Ax_1,Ax_0,a}(t)\left(\frac{t}{kn}\right)\quad \text{for } a \text{ in } X \text{ and } t>0. \tag{1}$$

We will now prove by induction that

$$F_{Ax_0,Ax_1,Ax_m}=(t)=1 \text{ for } m\ 0,\ 1,\ 2,\ \ldots \text{ and } t>0. \tag{2}$$

This is clearly rue when $m=0$ and $m=1$.
Now let us assume that equation (2) holds for all m with $0\leqslant n\leqslant l-1$.

By using axiom (F4) of definition 1 and inequality (1) we now obtain

$$\begin{aligned}&F_{Ax_0,Ax_1,Ax_k}(t)\geqslant\\&\geqslant \mathrm{Min}\left\{F_{Ax_0,Ax_1,Ax_{k-1}}\left(\frac{t}{3}\right),\ F_{Ax_1,Ax_0,Ax_0}\left(\frac{t}{3c^{k-1}}\right),\ F_{Ax_1,Ax_0,Ax_1}\left(\frac{t}{3c^{k-1}}\right)\right\}\end{aligned}$$

by our assumption and, consequently, the equation (2) follows by induction for $m=0,\ 1,\ 2,\ \ldots$

Using the inequality (1) we have

$$F_{Ax_{m+1},Ax_m,Ax_n}(t)\geqslant F_{Ax_1,Ax_0,Ax_n}\left(\frac{t}{k^m}\right)$$

and, consequently, if follows that $F_{Ax_{m+1}, Ax_m, Ax_n}(t) = 1$ for all $t > 0$ and for all $m, n = 0, 1, \ldots$ Thus, if $m < m+p = n$ and for any a in X, we have

$$F_{Ax_m, Ax_{m+p}, a}(t) \geqslant \operatorname{Min}\left\{F_{Ax_m, Ax_{m+1}, a}\left(\frac{t}{3}\right), F_{Ax_{m+1}, Ax_{m+p}, a}\left(\frac{t}{3}\right)\right\} \geqslant$$

$$\geqslant \operatorname{Min}\left\{F_{Ax_0, Ax_1, a}\left(\frac{t}{3k^m}\right), F_{Ax_{m+1}, Ax_{m+p}, a}\left(\frac{t}{3}\right)\right\} \geqslant$$

$$\geqslant \operatorname{Min}\left\{F_{Ax_0, Ax_1, a}\left(\frac{t}{3k^m}\right), F_{Ax_0, Ax_1, a}\left(\frac{t}{3^2 k^{m+1}}\right) \cdot \right.$$

$$\left. \cdot F_{Ax_0, Ax_1, a}\left(\frac{t}{3^p k^{m+p-1}}\right)\right\} \geqslant$$

$$\geqslant F_{Ax_0, Ax_1, a} \operatorname{Min}\left[\frac{t}{3k^m}, \frac{t}{3^2 k^{m+1}}, \ldots, \frac{t}{3^p k^{m+p-1}}\right].$$

Since $0 < k < 1$, if follows that

$$\lim_{m \to \infty} F_{Ax_m, Ax_{m+p}, a}(t) = 1 \qquad \text{for all } t > 0 \text{ and } p = 0, 1, 2, \ldots$$

Hence, $\{Ax_n\}$ is a Cauchy sequence with a limit u in X since X is complete. As $\{Sx_{2n+1}\}$ and $\{Tx_{2n}\}$ are subsequences of $\{Ax_n\}$, then also converge to some limit u.

Since A commutes with S and T, we have

$$F_{RAx_{2n}, SAx_{3n+1}, a}(t) = F_{ARx_{2n}, ASx_{2n+1}, a}(t) \geqslant$$

$$\geqslant \operatorname{Min}\left\{F_{ARx_{2n}, SRx_{2n}, a}\left(\frac{t}{k}\right), F_{SRx_{2n}, RSx_{2n+1}, a}\left(\frac{t}{k}\right)\right\}$$

and since S and R are continuous, on letting that n tends to infinity, we have

$$F_{Ru, Su, a}(t) \geqslant \operatorname{Min}\left\{F_{Ru, Su, a}\left(\frac{t}{k}\right), F_{Su, Ru, a}\left(\frac{t}{k}\right)\right\} = F_{Ru, Su, a}\left(\frac{t}{k}\right)$$

for all a in X and $t > 0$.

It follows immediately that

$$Ru = Su.$$

Now,

$$F_{RAx_{2n},\,An,\,a}(t)=F_{ARx_{2n},\,An,\,a}(t)\leqslant$$

$$\leqslant \operatorname{Min}\left\{F_{ARx_{2n},\,SRx_{2n},\,a}\left(\frac{t}{k}\right),\ F_{SRx_{2n},\,Rn,\,a}\left(\frac{t}{k}\right),F_{An,\,Rn,\,a}\left(\frac{t}{k}\right)\right\}$$

and on letting that n tends to infinity we have

$$F_{Ru,\,Au,\,a}(t)\geqslant \operatorname{Min}\left\{F_{Ru,\,Su,\,a}\left(\frac{t}{k}\right),\ F_{Su,\,Ru,\,a}\left(\frac{t}{k}\right),\ F_{Au,\,Ru,\,a}\left(\frac{t}{k}\right)\right\}=F_{Au,\,Ru,\,a}\left(\frac{t}{k}\right)$$

for all a in X and $t>0$.

Thus,

$$Au=Tu=Su.$$

We now have

$$F_{Au,\,A^2u,\,a}(t)\geqslant \operatorname{Min}\left\{F_{Au,\,Su,\,a}\left(\frac{t}{k}\right),F_{Su,\,RSu,a}\left(\frac{t}{k}\right),F_{A^2u,\,RAu,\,a}\left(\frac{t}{k}\right)\right\}=$$

$$=F_{Au,\,A^2u,\,a}\left(\frac{t}{k}\right)$$

for all a in X and $t>0$, and so

$$Au=A^2u.$$

Putting $Au=z$, we have

$$Az=A^2u=z=ASu=Sz=ARu=RAu=Rz.$$

Thus, $z\in K$ is a common fixed point of A, S and R. This proves the sufficiency of the condition.

To show that $x\in X$ is the unique common fixed point of A, S and R, let us suppose that $\omega\in X$ is the second common fixed point of A, S, R. Then,

$$F_{z,\,w,\,a}(t)=F_{Az,\,Az,\,a}(t)\geqslant$$

$$\geqslant \operatorname{Min}\left\{F_{Az,\,Sz,\,a}\left(\frac{t}{k}\right),\ F_{Sz,\,Rw,\,a}\left(\frac{t}{k}\right),\ F_{Aw,\,Rw,\,a}\left(\frac{t}{k}\right)\right\}=$$

$$=F_{z,\,w,\,a}\left(\frac{t}{k}\right)$$

for all a in X and $t>0$.

It follows that $z=w$, proving the uniqueness of z. This completes the proof of the theorem.

Remark 3. Every 2-metric space (X, d) may by regarded as an 2-PM space (see remark 2).

Then, putting $F_{xyz}(t)=\varepsilon(t-d(x, y, z))$, we have the following results: theorem 1 and 2 of Kubiak [3].

Taking $A=B$ and replacing the condition (1) by the following stronger condition

$$d(Ax, Ay, a)\leqslant c\cdot\max(d(Ax, Sx, a), d(Sx, Ty, a), d(Ay, Ty, a))$$

for all $x, y\in X$, $0<c<1$, we obtain theorem 2 of Khan and Fisher [2]. Let us also note that the assumption at continuity of A in theorem 2 of [2] is superfluous.

Theorem 1 of Singh-Tiwani-Gupta [6] is a special case of our theorem 1 if $A=B$ and $S=T$.

With $S=T=1_x$, the identity mapping of X theorem 1 reduces itself to theorem 4 of Roades [4].

References

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RELACJE PODWOJENIA

STRESZCZENIE

Pojęcie przestrzeni 2-metrycznych wprowadzone przez S. Gählen w pracy [1] jest naturalną abstrakcją funkcji pola trójkąta w przestrzeni Euklidesa.

W pracy tej uogólnia się twierdzenia o punktach stałych z przestrzeni 2-metrycznych ([4], [5]) na probabilistyczne przestrzenie 2-metryczne.