ON THE EXTENSION OF QUANTUM LOGIC STATES

FERDINAND CHOVANEC, FRANTIŠEK KÔPKA

A usual mathematical model for the quantum statistical mechanics is the quantum logic theory, i.e. the theory of orthomodular lattices ([5], [6]).

A state m on an orthomodular σ-complete lattice $L(\vee, \wedge, \perp, 0, 1)$ is a mapping m: $L \to [0, 1]$ ([0, 1] is the unit interval in the real line) satisfying the following two conditions:

1. $m(1)=1$

2. if $a_i \leqq a_j^{\perp}$ $(i \neq j)$, then $m(\bigvee_{i=1}^{\infty} a_i) = \sum_{i=1}^{\infty} m(a_i)$.

A. Dvurečenskij and B. Riečan pointed out ([1], [2]) that the same algebraic structure has the Piasecki P – measure ([3], [4]) and they introduced a new model for the statistical quantum mechanics. The Piasecki measure m: $M \to [0, 1]$ is defined on an appropriate set of real functions $M \subset [0, 1]^X$ and satisfies the following two conditions:

1. $m(f \vee f^{\perp})=1$ for every $f \in M$,

2. if $f_i \leqq f_j^{\perp}$ $(i \neq j)$, then $m(\bigvee_{i=1}^{\infty} f_i) = \sum_{i=1}^{\infty} m(f_i)$.

Of course, here $f^{\perp}=1-f$.

In the present paper we work with the mapping m: $A \to [0, 1]$ where A belongs to a class of lattices and in this way we obtain a common generalization of a state on a quantum logic as well as the Piasecki measure. The main result is a measure extension theorem. Special cases of our extension theorem is theorem on extension of states on logics [5] and the theorem on extension of the Piasecki measure [4].

Some definitions and notations.

We shal assume that a lattice H is given, satisfying the following conditions. Let H be a σ-continuous lattice with 0 and 1. Let $\perp : H \to H$ be a mapping satisfying conditions:

1.1. $(x^{\perp})^{\perp}=x$ for every $x \in H$,

1.2. if $x \leqq y$, then $y^{\perp} \leqq x^{\perp}$.

Let A be a sublattice of the lattice H such that

2.1. if $a, b \in A$, then $a \vee b \in A$,

2.2. if $a \in A$, then $a^{\perp} \in A$,

2.3. for every $x \in H$ there is sequence $(a_n)_{n=1}^{\infty}$, $a_n \in A$ $(n=1, 2,...)$, such that $x \leqq \bigvee_{n=1}^{\infty} a_n$.

There is given a mapping $m: A \to [0, 1]$ satisfying the following conditions:

3.1. $m(a \vee a^{\perp}) = 1$ for every $a \in A$,

3.2. if $a_i \in A$ $(i=1, 2, ...)$, $a_i \leqq a_j^{\perp}$ $(i \neq j)$, $\bigvee_{i=1}^{\infty} a_i \in A$, then $m(\bigvee_{i=1}^{\infty} a_i) = \sum_{i=1}^{\infty} m(a_i)$,

3.3. m is a strong continuous mapping (i.e. if $a_n \searrow a$, $b_n \nearrow b$, $a_n, b_n \in A$, $a \leqq b$, $a, b \in H$, then $m(a_n \wedge b_n^{\perp}) \searrow 0$),

3.4. m is a valuation (i.e. $m(a \vee b) + m(a \wedge b) = m(a) + m(b)$).

It is easy to prove that m has the following properties:

4.1. $m(a) + m(a^{\perp}) = 1$ for every $a \in A$,

4.2. if $a, b \in A$, $a \leqq b$, then $m(b) = m(a) + m(b \wedge a^{\perp})$,

4.3. m is non – decreasing,

4.4. m is upper continuous (i.e. if $b_n \nearrow b$, $b_n, b \in A$, then $m(b_n) \nearrow m(b)$),

4.5. m is subadditive.

We want to extend the map m on the smolest σ-complete lattice $S(A)$, generated by the sublattice A. Our extension will be made in the standard way.

Let $A^+ = \{b \in H; \exists\ b_n \nearrow b, b_n \in A\}$ and $m^+ : A^+ \to [0, 1]$, $m^+(b) = \lim_{n \to \infty} m(b_n)$.

Moreover definition of m^+ is correct, due to the following assertion.

LEMMA 1. Let $a_n, b_n \in A$ $(n=1, 2, ...)$, $a_n \nearrow a$, $b_n \nearrow b$, $a \leqq b$. Then $\lim_{n \to \infty} m(a_n) \leqq \lim_{n \to \infty} m(b_n)$.

Proof. Evidently $a_n \wedge b_k \nearrow a_n \wedge b = a_n$ $(k \to \infty)$, hence

$$m(a_n) = \lim_{k \to \infty} m(a_n \wedge b_k) \leqq \lim_{k \to \infty} m(b_k)$$

and therefore

$$\lim_{n \to \infty} m(a_n) \leqq \lim_{k \to \infty} m(b_k).$$

It is not dificult to prove that m^+ is an extension of m, upper continuous, non – decreasing and m^+ is a valuation.

Let us define $m^*: H \to [0, 1]$ in the following way:

$$m^*(x) = \inf\{m^+(b); b \in A^+, b \geqq x\}.$$

It is easy to see, that m^* is an axtension of m, m^+; non – decreasing; for every $x, y \in H$ $m^*(x \vee y) + m^*(x \wedge y) \leqq m^*(x) + m^*(y)$ and therefore the map is subadditive.

Similarly we can define $A^- = \{c \in H: \exists\ c_n \searrow c, c_n \in A\}$,

$$m^-: A^- \to [0, 1], \quad m^-(c) = \lim_{n \to \infty} m(c_n),$$

$$m_*: H \to [0, 1], \quad m_*(x) = \sup\{m^-(c): c \in A^-, c \leqq x\}.$$

The straightforvard consequence of the above mentioned definitions of the

m^-, m_* are the following properties: m^- is an extension of m, a non – decreasing map and a valuation; m_* is an extension of m^-, a non – decreasing map, for every $x, y \in H$

$$m_*(x \vee y) + m_*(x \wedge y) \geqq m_*(x) + m_*(y).$$

The following theorem is the main assertion of this paper.

THEOREM. Let H be a σ-contiuous lattice with 0 and 1, which satisfies the conditions 1.1. and 1.2. Let A be a sublattice of the lattice H which satisfies the conditions 2.1, 2.2, 2.3. Let $m: A \to [0, 1]$ be the map satisfing the conditions 3.1 to 3.4. Let $S(A)$ be the least σ-complete sublattice of the lattice H generated by the sublattice A. Then there exists exactly one map $\bar{m}: S(A) \to [0, 1]$ that is an extension of m and satisfies 3.1 and 3.2. Moreover $\bar{m}$ is a valuation and a strong contiunous map.

Before the proving of the above mentioned theorem we shall introduce some helpful lemmas.

Let us construct the set $L = \{x \in H: m_*(x) = m^*(x)\}$. The main idea lies in proving $L \supset S(A)$ and in the fact that $m^*/S(A)$ is the searched extension.

LEMMA 2. Let $b \in A^+$, then $b^\perp \in A^-$ and $m^+(b) + m^-(b^\perp) = 1$.

Proff. Let $b_n \nearrow b$, $b_n \in A$, then $b^\perp \leqq b_n^\perp$, $b^\perp \leqq \bigwedge_{n=1}^{\infty} b_n^\perp = c$. Since $c \leqq b_n^\perp$, $b_n \leqq c^\perp$, we obtain $b = \bigvee_{n=1}^{\infty} b_n \leqq c^\perp$, $c \leqq b^\perp$. Then $c = b^\perp$ and $b^\perp \in A^-$. By 4.1 we have $m(b_n) + m(b_n^\perp) = 1$. Taking $n \to \infty$ we obtain $\lim_{n\to\infty} m(b_n) + \lim_{n\to\infty} m(b_n^\perp) = 1$, i.e. $m^+(b) + m^-(b^\perp) = 1$.

PROPOSITION 1. If $x \in H$, then $m_*(x) \leqq m^*(x)$.

Proof. Let $x \in H$, $a \in A^-$, $b \in A^+$, $a \leqq x \leqq b$. Then

$$m(b_n \vee a_n^\perp) = m((b_n^\perp \wedge a_n)^\perp) = 1 - m(b_n^\perp \wedge a_n).$$

By strong contiunous and subaditive m we have

$$1 = \lim_{n\to\infty} m(b_n \vee a_n^\perp) \leqq \lim_{n\to\infty} m(b_n) + \lim_{n\to\infty} m(a_n^\perp) = m^+(b) + m^+(a^\perp).$$

Hence $1 - m^+(a^\perp) = m^-(a) \leqq m^+(b)$. Then $m_*(x) \leqq m^*(x)$, too.

PROPOSITION 2. Let $x, y \in L$, $y \leqq x$. Then $m^*(x) = m^*(y) + m^*(x \wedge y^\perp)$.

Proof. 1. Let $a \in A$, $b \in A^+$, $a \leqq b$. Then $m^+(b) = m(a) + m^+(b \wedge a^\perp)$. Since $a_n \nearrow b$, $a_n \wedge a^\perp \nearrow b \wedge a^\perp$, we obtain

$$m^+(b) = m(a) + m^+(b \wedge a^\perp).$$

2. Let $b, d \in A^+$, $d \leqq b$. Then $m^+(b) \geqq \xi^+(\delta) + \xi^*(\beta \wedge \delta^\perp)$.

Indeed, $d_n \nearrow d$, $d_n \in A$ and 1. imply

$$m^+(b) = m(d_n) + m^+(b \wedge d_n^\perp) \geqq m(d_n) + m^*(b \wedge d^\perp),$$

wich gives

$$m^+(b) \geqq m^+(d) + m^*(b \wedge d^\perp).$$

3. Let $b \in A^+$, $c \in A^-$, $c \leqq b$. Then $m^+(b) \geqq m^-(c) + m^+(b \wedge c^\perp)$. Take $c_n \in A$, $c_n \searrow c$. Since $b \wedge c_n \in A^+$, $b \wedge c_n \leqq b$ we have by 2.

$$m^+(b) \geqq m^+(b \wedge c_n) + m^*(b \wedge (b \wedge c_n)^\perp) \geqq m^+(b \wedge c_n) + m^*(b \wedge c_n^\perp) =$$

$$= m^+(b \wedge c_n) + m^+(b \wedge c_n^\perp).$$

Taking $n \to \infty$ we obtain

$$m^+(b) \geqq \lim_{n\to\infty} m^+(b \wedge c_n) + \lim_{n\to\infty} m^+(b \wedge c_n^\perp) \geqq m^-(c) + m^+(b \wedge c^\perp).$$

4. Let $x \in L$, $c \in A^-$, $c \leqq x$. Then $m^*(x) \geqq m^-(c) + m^*(x \wedge c^\perp)$.

Namely, if $b \in A^+$, $b \geqq x$, then

$$m^+(b) \geqq m^-(c) + m^+(b \wedge c^\perp) \geqq m^-(c) + m^*(x \wedge c^\perp),$$

hence

$m^*(x) \geqq m^-(c) + *(x \wedge c^\perp)$, too.

5. Finally we prove the assertion stated in Proposition. Let $x, y \in L$, $y \leqq x$. Take $c \in A^-$, $c \leqq y$. By 4. we have

$$m^*(x) \geqq m^-(c) + m^*(x \wedge c^\perp) \geqq m^-(c) + m^*(x \wedge y^\perp),$$

hence

$$m^*(x) - m^*(x \wedge y^\perp) \geqq m^-(c).$$

Therefore $m^*(x) - m^*(x \wedge y^\perp) \geqq m_*(y) = m^*(y)$.

We obtain the opposite inequality in the following way. Let $c \in A^-$, $c \leqq x$ and $b \in A^+$, $b \geqq y$. Then $b^\perp \in A^-$, $b^\perp \wedge c \in A^-$. It is valid that

$$1 = m^-(b^\perp \wedge c) + m^+(b \vee c^\perp) \leqq m^-(b^\perp \wedge c) + m^+(b) + m^+(c^\perp).$$

Then

$$m^-(c) = 1 - m^+(c^\perp) \leqq m^-(b^\perp \wedge c) + m^+(b) \leqq m_*(x \wedge y^\perp) + m^+(b).$$

Next

$$m_*(x) \leqq m_*(x \wedge y^\perp) + m^*(y) \leqq m^*(x \wedge y^\perp) + m^*(y).$$

Because $x \in L$ we obtain $m^*(x) \leqq m^*(y) + m^*(x \wedge y^\perp)$.

LEMMA 3. Let $x \in L$. Then $m^*(x) + m^*(x^\perp) = 1$.

Proof. Take $a \in A^+$, $a \geqq x^\perp$. Then $a^\perp \in A^-$, $a^\perp \leqq x$. From the inequality $m_*(x) \geqq m^-(a^\perp) = 1 - m^+(a)$ it follows $m^*(x) + m^+(a) \geqq 1$ for every $a \in A^+$, $a \geqq x^\perp$ and then $m^*(x) + m^*(x^\perp) \geqq 1$. Let $c \in A^-$, $c \leqq x$. Then $c^\perp \geqq x^\perp$, $c^\perp \in A^+$ and

$$1 = m^-(c) + m^+(c^\perp) \geqq m^-(c) + m^*(x^\perp),$$

hence

$$1 \geqq m_*(x) + m^*(x^\perp) = m^*(x) + m^*(x^\perp).$$

PROPOSITION 3. Let $x \in L$. Then $x^{\perp} \in L$.

Proof. Take $a \in A^{+}$, $a \geqq x$. Then $a^{\perp} \in A^{-}$, $a^{\perp} \leqq x^{\perp}$ and then $m^{-}(a^{\perp}) \leqq m_{*}(x^{\perp})$, i. e. $1 - m^{+}(a) \leqq m_{*}(x^{\perp})$, $1 \leqq m^{+}(a) + m_{*}(x^{\perp})$.

Because a is arbitrary element from A^{+}, we have

$$1 \leqq m^{*}(x) + m_{*}(x^{\perp}).$$

On the other hand

$$1 = m^{*}(x) + m^{*}(x^{\perp}) \leqq m^{*}(x) + m_{*}(x^{\perp}).$$

The equality $m_{*}(x) = m^{*}(x)$ implies $m^{*}(x^{\perp}) \leqq m_{*}(x^{\perp})$, i.e. $x^{\perp} \in L$.

PROPOSITION 4. Let $x_n \in L\,(n = 1, 2, \ldots)$, $x_n \nearrow x$ (or $x_n \searrow x$ resp.), $x \in H$. Then $x \in L$ and $m^{*}(x) = \lim_{n \to \infty} m^{*}(x_n)$.

Proof. If $x_n \in L$, then for every $\varepsilon > 0$ exist $b_n \in A^{+}$, $b_n \geqq x_n$ $(n = 1, 2, \ldots)$, that $m^{*}(x_n) + \frac{\varepsilon}{2^n} > m^{+}(b_n)$. Take $u_n = \bigvee_{i=1}^{n} b_i$. Then $u_n \in A^{+}$, $u_n \nearrow \bigvee_{n=1}^{\infty} u_n \geqq \bigvee_{n=1}^{\infty} x_n = x$. The valuation m^{+} implies

$$m^{+}(u_2) = m^{+}(b_1 \vee b_2) = m^{+}(b_1) + m^{+}(b_2) - m^{+}(b_1 \wedge b_2) \leqq$$

$$\leqq m^{+}(b_1) - m^{*}(x_1) + m^{+}(b_2) < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + m^{*}(x_2).$$

By induction we prove $m^{+}(u_n) < m^{*}(x_n) + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \ldots + \frac{\varepsilon}{2^n}$. Next

$$m^{*}(x) \leqq m^{+}\Big(\bigvee_{n=1}^{\infty} u_n\Big) = \lim_{n \to \infty} m^{+}(u_n) \leqq \lim_{n \to \infty} m^{*}(x_n) + \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} =$$

$$= \lim_{n \to \infty} m^{*}(x_n) + \varepsilon, \quad \text{i.e. } m^{*}(x) \leqq \lim_{n \to \infty} m^{*}(x_n).$$

We obtain the oposite inequality from non – decreasing m^{*}. Further

$$m_{*}(x) \leqq m^{*}(x) = \lim_{n \to \infty} m^{*}(x_n) = \lim_{n \to \infty} m_{*}(x_n) \leqq m_{*}(x), \quad \text{hence } x \in L.$$

The second part of Proposition 4 (for non – increasing sequences) follows from Proposition 3.

LEMMA 4. Let A be a sublattice of the lattice H wich follows the conditions 2.1, 2.2, 2.3. Let $S(A)$ be the least δ- complete sublattice of the lattice H generated by the sublattice A. Let $M(A)$ be the least set over A closed under monotone sequences. Then $S(A) = M(A)$.

Proof. Since $S(A)$ is monotone set, it is $M(A) \subset S(A)$. Let $x \in A$. Put $G = \{y \in M(A) : x \vee y \in M(A)\}$. Evidently $A \subset G$, G is monotone set, therefore $M(A) \subset G$. So, $x \vee y \in M(A)$ for every $y \in M(A)$, $x \in A$. Take $y \in M(A)$ now and put $K = \{x \in M(A) : x \vee y \in M(A)\}$. We have $A \subset K$, K is monotone set.

We get $M(A) \subset K$ and so $x \vee y \in M(A)$ for every $x, y \in M(A)$. Similarly we can prove, that for every $x, y \in M(A)$, $x \wedge y \in M(A)$, too. Let N be a set $N = \{x \in M(A): x^{\perp} \in M(A)\}$. We see that $N \subset M(A)$, $A \subset N$, N is monotone set, therefore $M(A) \subset N$. We have $N = M(A)$ and so $x^{\perp} \in M(A)$ for every $x \in M(A)$, i.e. $M(A)$ is a sublattice of the lattice H. *Then* $S(A) \subset M(A)$.

PROPOSITION 5. Let $x, y \in S(A)$, $x \leqq y^{\perp}$. Then $m^*(x \vee y) = m^*(x) + m^*(y)$.

Proof. Since L is a monotone set, it is $M(A) = S(A) \subset L$. If $x, y \in S(A)$, $x \leqq y^{\perp}$, then $x^{\perp} \wedge y^{\perp} \in S(A)$ and by means of Proposition 2. we get

$$m^*(y^{\perp}) = m^*(x) + m^*(y^{\perp} \wedge x^{\perp}) = m^*(x) + 1 - m^*(y \vee x).$$

Hence

$$m^*(x \vee y) = m^*(x) + 1 - m^*(y^{\perp}) = m^*(x) + m^*(y).$$

Proof ot Theorem.

1. Existence. Put $\bar{m} = m^*/S(A)$. According to Proposition 5 and Lemma 3 we have

$$\bar{m}(x \vee x^{\perp}) = \bar{m}(x) + \bar{m}(x^{\perp}) = 1 \qquad \text{for every } x \in S(A).$$

According to Proposition 4 and 5 $\bar{m}$ is upper continuous and aditive. We shall prove, that $\bar{m}$ is σ-aditive. Let $(x_n)_{n=1}^{\infty}$ be a sequence, $x_n \in S(A)$ $(n = 1, 2, \ldots)$ and x_n are pairwise orthogonal (i.e. $x_i \leqq x_j^{\perp}$ for $i \neq j$).

Take $a_n = \bigvee_{i=1}^{n} x_i$, $a_n \in S(A)$ $(n = 1, 2, \ldots)$. Then $\bar{m}(a_h) = \sum_{i=1}^{n} \bar{m}(x_i)$.

Evidently

$$\bigvee_{n=1}^{\infty} a_n = \bigvee_{n=1}^{\infty} \left(\bigvee_{i=1}^{n} x_i\right) = \bigvee_{i=1}^{\infty} x_i \in S(A),$$

$$\bar{m}\left(\bigvee_{i=1}^{\infty} x_i\right) = \bar{m}\left(\bigvee_{n=1}^{\infty} a_n\right) = \lim_{n=\infty} \bar{m}(a_n) = \lim_{n \to \infty} \bar{m}\left(\bigvee_{i=1}^{n} x_i\right) =$$

$$= \lim_{n-\infty} \sum_{i=1}^{n} \bar{m}(x_i) = \sum_{i=1}^{\infty} \bar{m}(x_i).$$

For every $x, y \in S(A) \subset H$ the following inequalities hold

$$m^*(x \vee y) + m^*(x \wedge y) \leqq m^*(x) + m^*(y)$$

and

$$m_*(x \vee y) + m_*(x \wedge y) \geqq m_*(x) + m_*(y).$$

Since $S(A) \subset L$, we can rewrite the last inequality as

$$m^*(x \vee y) + m^*(x \wedge y) \geqq m^*(x) + m^*(y),$$

so that the mapping $\bar{m} = m^*/S(A)$ is a valuation.

Let $x_n \searrow x$, $y_n \nearrow y$, $x_n, y_n \in S(A)$ $(n = 1, 2, \ldots)$, $x \leqq y$. Then $x, y \in S(A)$ and for every n: $\bar{m}(x_n \vee y_n^{\perp}) + \bar{m}(x_n \wedge y_n^{\perp}) = \bar{m}(x_n) + \bar{m}(y_n^{\perp})$. Evidently

$x_n \vee y_n^\perp \searrow x \vee y^\perp$ and $\lim_{n\to\infty} \bar{m}(x_n \vee y_n^\perp = \bar{m}(x \vee y^\perp) = \bar{m}(x) + \bar{m}(y^\perp)$. For $n \to \infty$ we obtain equality

$$\lim_{n\to\infty} \bar{m}(x_n \vee y_n^\perp) + \lim_{n\to\infty} \bar{m}(x_n \wedge y_n^\perp) = \bar{m}(x) + \bar{m}(y^\perp)$$

and hence

$$\lim_{n\to\infty} (\bar{m}(x_n \wedge y_n^\perp) = 0 .$$

Uniqueness. Let $m_1 : S(A) \to [0, 1]$ be a map, $m_1/A = m$. Put $G = \{x \in S(A) : \bar{m}(x) = m_1(x)\}$. *Evidently* $A \subset G$, $G \subset S(A)$, G is closed under limits of monotone sequences. Therefore $G \supset M(A) = S(A)$. Then $\bar{m}/S(A) = m_1/S(A)$.

References

[1] A. Dvurečenskij, B. Riečan, On joint observables for F-quantum spaces. Fuzzy sets and systems, to appear.
[2] A. Dvurečenskij, B. Riečan, On randomness and fuzziness. In: Progress in fuzzy sets in Europe.
[3] K. Piasecki, Probability of fuzzy events defined as denumerable additivity measure. Fuzzy sets and systems 17 (1985), 271-284.
[4] K. Piasecki, Extension of fuzzy P-measure. Busefal 19 (1984), 26-41.
[5] B. Riečan, The measure extension theorem for subadditive probability measures in orthomodular σ-continuous lattices. Comment. Math. Univ. Carol. 2o, 2 (1979), 309-315.
[6] V. S. Varadarajan, Geometry of quantum theory. Van Nostrand, New York 1968.