A NOTE ON SOME INTERESTING RESULTS IN FUZZY LOGIC

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1. Introduction

In this paper, we will discuss some of the results in many-valued fuzzy logic, i.e. the logic with more than two truth values which are crisp.

One of the greatest problems in fuzzy logic is too extensive variety of possibilities how to introduce its syntax and semantics. Hence, we can obtain many systems with various properties more or less fullfilling our intuition.

In 1979 J. Pavelka published three papers [4] which put much light on the problem of proper choice of a system of fuzzy logic. The main demand is fuzzy logic to be a direct generalisation of classical one in such a way that it should preserve as much properties of classical logic as possible and become classical one in case we confine the system to two truth values only. Pavelka demonstrated that such fuzzy logic must be based on either finite chain or the interval [0, 1] of reals serving as the set of truth values. Morever, he introduced two more basic operations (in addition to the supremum $\vee$ and infinum $\wedge$), namely the bold product $\otimes$ and residuum $\rightarrow$ and demonstrated that this choice is the only possible one (up to an isomorphism). He proved some important properties of fuzzy logic. One may ask whether is it possible to extend this system also to first-order fuzzy logic in such a way that it will have also such good properties as propositional fuzzy logic has. In this paper, some results proved by the author are presented.

2. Few remarks on syntax and semantics

A structure of truth values considered is a residuated lattice

$$\mathscr{L}=[\mathsf{L}, \vee, \wedge, \oplus, \rightarrow, \mathbf{1}, \mathbf{0}]$$

where $\mathsf{L}=[0, 1]$ or it is a finite chain, $\vee$, $\wedge$ are the operations of supremum and infimum respectively.

$$\alpha \otimes \beta = 0 \vee (\alpha + \beta - 1)$$

is bold product and

$$\alpha \to \beta = 1 \wedge (1 - \alpha + \beta)$$

is residuum, $\alpha, \beta \in [0, 1]$ (analogous definition of $\otimes$, $\to$ proceeds in the case of $\mathbb{L}$ being a finite chain – see [1], [2], [3], [4]). The $\mathscr{L}$ can be extended by new additional operations. For discussion of this problem see [3].

Terms in fuzzy logic are only variables or constants. Formulas are derived from atomic formulas using the connectives $\wedge$ (conjunction), $\vee$ (disjunction), & (bold conjunction), $\Rightarrow$ (implication), and quantifiers $\forall$, $\exists$ as usual.

Let us note that atomic formulas are also symbols for truth values α, $\alpha \in L$. Moreover, all the connectives can be defined using only $\Rightarrow$ and $\mathbf{0}$. The negation is a short for the formula

$$\varphi \Rightarrow \mathbf{0}.$$

There are three inference rules:

Modus ponens

$$r_{MP}: \frac{\varphi, \varphi \Rightarrow \psi}{\psi} \left(\frac{\alpha, \beta}{\alpha \otimes \beta} \right),$$

α – lifting rule

$$r_{R\alpha}: \frac{\varphi}{\alpha \Rightarrow \varphi} \left(\frac{\beta}{\alpha \to \beta} \right).$$

Generalization

$$r_G: \frac{\varphi}{(\forall x)\varphi} \left(\frac{\alpha}{\alpha} \right),$$

$\alpha, \beta \in L$. In the bracket, the way how to compute the truth value of the resulting formula from the truth values of the respective formulas entering the rule is presented (semantic part of the rule).

A theory $\mathscr{T}$ is a three-tuple

$$\mathscr{T} = \langle A_L, A_S, \mathscr{R} \rangle$$

where A_L, A_S are fuzzy sets of logical and special axioms respectively and $\mathscr{R}$ is the set of rules of inference.

A formal proof is any sequence of the form

$$w : \varphi_0 [\alpha_0; P_0], \ldots, \varphi_n [\alpha_n; P_n]$$

where P_i is either LA (SA) if φ_i is a logical (special) axiom (then α_i is the membership degree $A_L \varphi_i$ or $A_S \varphi_i$ respectively) or it is r, if φ_i is derived from some preceeding formulas φ_j, $j = 1, \ldots, m$ using the m-ary rule of inference $r \in \mathscr{R}$ (then α_i is computed from α_j, $1, \ldots, m$; in our case, $m = 1$ or $m = 2$). We say that ω is a proof of the formula φ_n with the value

$$\mathrm{Val}(\omega) = \alpha_n.$$

Let $\mathscr{J} = \{p, q, \ldots : \mathbf{a}_1, \mathbf{a}_2, \ldots\}$ be a language of the first-order fuzzy theory where $p, q, \ldots$ are n-ary predicate symbols (n depends on $p, q, \ldots$) and $\mathbf{a}_i$ are

constants. A structure for $\mathscr{J}$ is

$$\mathscr{D}=\langle D;\ p_{\mathscr{D}}, q_{\mathscr{D}}, \ldots ;\ a_1,\ a_2, \ldots\rangle$$

where D is a set $p_{\mathscr{D}} \subseteq \mathscr{D}^n$, ... are fuzzy relations assigned to p, ... and a_i are assigned to $\mathbf{a}_i$. Let $\mathscr{T}$ be a fuzzy theory. Then a structure $\mathscr{D}$ is a model of the theory $\mathscr{T}$, $\mathscr{D} \models \mathscr{T}$, if

$$\mathscr{D}(\varphi) \geqq A_S \varphi$$

holds for every formula φ. The symbol $\mathscr{D}(\varphi)$ denotes the truth value of the formula φ in the structure $\mathscr{D}$ (for details see [1], [2]).

The degree of provability of φ in the theory $\mathscr{T}$ is

$$\alpha=\bigvee\{\mathrm{Val}(\omega),\ \omega \text{ is a proof of } \varphi\}.$$

We write

$$\mathscr{T} \underset{\alpha}{\vdash} \varphi.$$

The degree of truth of φ in the theory $\mathscr{T}$ is

$$\beta=\bigwedge\{\mathscr{D}(\varphi);\ \mathscr{D} \text{ is a model of } \mathscr{T}\}.$$

We write

$$\mathscr{T} \underset{\beta}{\models} \varphi.$$

3. Some theorems

3.1 Consistency of a theory

A theory $\mathscr{T}$ is consistent if

$$\mathscr{T} \underset{\alpha}{\vdash} \varphi \text{ and } \mathscr{T} \underset{\beta}{\vdash} \neg\varphi \text{ implies } \alpha \otimes \beta = \mathbf{0}$$

for every formula φ. In the opposite case it is contradictory.

THEOREM 1. If a theory $\mathscr{T}$ has a model then it is consistent.

This theorem is quite analogous to the corresponding classical one. More surprising is that if $\mathscr{T} \underset{\alpha}{\vdash} \varphi$, $\mathscr{T} \underset{\beta}{\vdash} \neg\varphi$ and $\alpha \otimes \beta > \mathbf{0}$ then necessarily each formula is provable in the degree 1.

THEOREM 2. A theory is contradictory iff $\mathscr{T} \vdash \varphi$ for every formula φ.

It follows from the definition that we can find more proofs of the same formula which differ in their values. Thence, finding a proof ω of a formula φ with the value, say

$$\mathrm{Val}(\omega)=\beta,$$

means only that the formula φ is provable in the degree greater than or equal to β. Theorem 2 says that if we find proofs ω_φ and $\omega_{\neg\varphi}$ such that

$$\mathrm{Val}(\omega_\varphi) \otimes \mathrm{Val}(\omega_{\neg\varphi}) > \mathbf{0}$$

then supremum of values of all the proofs of any formula is equal to 1.

For each $\alpha \in \mathsf{L}$ and every theory $\mathscr{T}$

$$\mathscr{T} \vdash_{\alpha} \alpha.$$

If we find a proof of α with the value greater than α then $\mathscr{T}$ is contradictory.

3.2 Deduction theorem

Extension of theories is very important problem. Let B be a fuzzy set of formulas. Then the theory

$$\mathscr{T}' = \langle A_L, A_S \cup B, \mathscr{R} \rangle \mathscr{T} \cup B$$

is an extension of the theory $\mathscr{T}$. As a special case, B can be a fuzzy singleton $\{\alpha/\varphi\}$. It can be immediately seen that

$$\mathscr{T} \vdash_{\alpha} \varphi \text{ and } \mathscr{T}' \vdash_{\beta} \varphi \text{ implies } \alpha \leqq \beta.$$

A crucial role in classical logic plays deduction theorem stating

$$\mathscr{T} \cup \{\varphi\} \vdash \psi \text{ iff } \mathscr{T} \vdash \varphi \Rightarrow \psi.$$

A direct generalisation of this theorem does not hold in fuzzy logic. The reason is that the formula

$$\varphi \Rightarrow (\psi \Rightarrow \chi) \Rightarrow ((\varphi \Rightarrow \psi) \Rightarrow (\varphi \Rightarrow \chi))$$

is not a tautology (in the degree 1). Fortunately, a weaker form of deduction theorem can be proved. By φ^n we denote the formula

$$\underbrace{\varphi \,\&\, \varphi \,\&\, \ldots \,\&\, \varphi}_{n \text{ - times}}.$$

THEOREM 3 (deduction theorem). Let φ be a closed formula.

a) Let $\mathscr{T}' = \mathscr{T} \cup \{1/\varphi\}$. If $\mathscr{T} \vdash_{\delta} \varphi^n \Rightarrow \psi$ and $\mathscr{T}' \vdash_{\delta'} \psi$ for some n then $\delta \leqq \delta'$.
b) Let $\mathscr{T}' = \mathscr{T} \cup \{\alpha/\varphi\}$. To every proof ω'_{ψ} in $\mathscr{T}'$ there are n and a proof $\omega_{\varphi^n \Rightarrow \psi}$ in $\mathscr{T}$ such that

$$\mathrm{Val}_{\mathscr{T}}(\omega'_{\psi}) \leqq \mathrm{Val}_{\mathscr{T}}(\omega_{\varphi^n \Rightarrow \psi})$$

COROLLARY. Let L be a finite chain and $\mathscr{T}' = \mathscr{T} \cup \{1/\varphi\}$. Then

$$\mathscr{T} \vdash_{\delta} \varphi^n \Rightarrow \psi \text{ iff } \mathscr{T}' \vdash_{\delta} \psi$$

for some n.

It can be immediately seen that this theorem becomes classical deduction theorem if we confine curselves only to $\mathsf{L} = \{0, 1\}$. So far, it si not known if corollary holds also for $\mathsf{L} = [0, 1]$. The demonstration would have to de based on special topological properties of the proofs with values in $[0, 1]$. However, it is sure that we cannot replace "some n" by $n = 1$ as in the classicall theorem. The core of this theorem consists in the fact that the operation $\otimes$ in L is nilpotent for every $\alpha < 1$.

The deduction theorem enables us to prove important theorem on Henkin extension of a theory which is necessary in the proof of the completeness theorem.

3.3. Completeness theorem

One of the most important theorems in classical logic is Gödel's completeness theorem:

$$\mathscr{T} \vdash \varphi \text{ iff } \mathscr{T} \models \varphi$$

for every formula φ. Pavelka proved this theorem for propositional fuzzy logic. The proof is difficult and uses the ultrafilter trick. Using deduction theorem and analogous trick it is possible to prove completeness theorem also in first-order fuzzy logic. It has two forms:

THEOREM 4. $\mathscr{T}$ be a consistent theory. Then

$$\mathscr{T} \vdash_{\alpha} \varphi \text{ iff } \mathscr{T} \models_{\alpha} \varphi$$

for every formula φ.

THEOREM 5. A fuzzy theory $\mathscr{T}$ is consistent iff it has a model. Let $\mathscr{T}$ be a consistent theory and $\mathscr{T} \vdash_{\alpha} \varphi$. Put $\beta = \alpha$ if L is finite and $\beta > \alpha$ if $L = [0, 1]$. Then there is a model $\mathscr{D} \models \mathscr{T}$ such that

$$\mathscr{D}(\varphi) \leqq \beta .$$

This theorem is not as strong as in classical logic. It only assures us that a consistent theory must have a model. However, if we find a model then we have not obtained so much since the truth value $\mathscr{D}(\varphi)$ is in general greater than the degree of provability of φ. From this theorem we know that the infimum of $\mathscr{D}(\varphi)$ in all the models is equal to the degree of provability of φ.

4. Conclusion

We see that the situation in fuzzy logic is more complicated than in classical one. It is possible to prove quite strong theorems but they are not as strong as analogous classical ones. An interesing fact is that these results can be proved also if we enrich fuzzy logic by new connectives, most of them being included in the other systems of fuzzy logic studies and, hence, hold in some sense also for them. More complicated situation appears if we introduce new generalised quantifiers. This problem needs still more investigation.

Refferences

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