# REMARKS ON A FUZZY FUNCTION AND ITS FIXED POINT

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In the literature one encounters various definitions of a fuzzy function. They will be quoted below.

Let us denote by $FS(U)$ the collection of all subsets in a space $U$.

DEFINITION 1. By a fuzzy function we mean a mapping $f$ of the form

$$f: FS(U_1) \to FS(U_2),$$

that is, the mapping assigning to each fuzzy set $A$ in a space $U_1$ ($A \in FS(U_1)$) a fuzzy set $B$ in a space $U_2$ ($B \in FS(U_2)$).

DEFINITION 2. By a fuzzy function we shall mean a mapping $f: U_1 \to U_2$ such that if $A$ is a fuzzy set in a space $U_1$, $B$ is a fuzzy set in a space $U_2$, whereas $R_A$, $R_B$ are the corresponding membership functions of these sets, then the condition

$$R_B(f(x)) \geqq R_A(x) \tag{1}$$

is satisfied.

The example below explains the functioning of this definition.

**Example** 1. We are given two spaces

$$U_1 = \{0,1,2,6,7,10\}, \quad U_2 = \{1,7,8,9,11\}$$

and a function $f: U_1 \to U_2$ defined as follows:

$$f(0)=1, \quad f(1)=7, \quad f(2)=8,$$
$$f(6)=0, \quad f(7)=9, \quad f(10)=11.$$

Let $A$ be a fuzzy set, contained in $U_1$, of numbers expressing the notion „several", whereas $B$ a fuzzy set of numbers of $U_2$ „close to the number 10". We shall define the membership functions for these sets in such a way that

$$R_A(0)=0, \quad R_A(1)=0, \quad R_A(2)=0,2,$$
$$R_A(6)=1, \quad R_A(7)=0,7, \quad R_A(10)=0,1$$

and

$$R_B(1)=0, \quad R_B(7)=0,5, \quad R_B(8)=0,8, \quad R_B(9)=R_B(11)=1.$$

We then have

| $x$ | $R_A(x)$ | $f(x)$ | $R_B(f(x))$ |
|---|---|---|---|
| 0 | 0 | 1 | 0 |
| 1 | 0 | 7 | 0,5 |
| 2 | 0,2 | 8 | 0,8 |
| 6 | 1 | 9 | 1 |
| 7 | 0,7 | 9 | 1 |
| 10 | 0,1 | 11 | 1 |

From the above juxtaposition it can be seen that $f$ is a fuzzy function in the sense of definition 2 since condition (1) is satisfied.

DEFINITION 3. (according to D. Butnariu [1]). Let X stand for an arbitrary family of sets.

By a fuzzy function we mean each mapping of the form

$$f: \mathrm{X} \to FS(\mathrm{X}),$$

i.e. the mapping assigning to each set $X \in \mathrm{X}$ a fuzzy set $X' \in FS(\mathrm{X})$.

This function, in contradistinction to the fuzzy functions described in the previous definitions, is defined on an arbitrary set X. Treating this function as a relation, we may identify it with a set of pairs from the Cartiesian product $\mathrm{X} \times FS(\mathrm{X})$, thus with the set $\{(X, X')\}$ where $X \in \mathrm{X}$, $X' \in FS(\mathrm{X})$.

Since a fuzzy function (in the sense given here) determines the fuzzy set $X'$ corresponding to a non-fuzzy set $X$, it is closely connected with the determining function of this set. The determining function does not indicate the set itself directly, but states the degree of the belonging of elements to this set. However, in computation practice, one may – when it is a question of a fuzzy function – use the determining function. This happens in the case of a fixed point, as well.

For the fuzzy mapping defined above, we adopt the following definition of a fixed point $x_0$.

DEFINITION 4. By a fixed point of a fuzzy mapping $f$ defined on an arbitrary set X we mean any element $x_0$ from a set $X$, such that

$$f(x_0, x_0) \geqq f(x_0, x) \quad \text{for any } x \in X. \tag{2}$$

We shall now examine whether there is a fixed point of a fuzzy mapping in the following example.

**Example 2.** Let $X = \{1, 2, 3, 4\}$. Let us consider the set of those pairs $(x, y)$, $x, y \in X$, for which the number $x$ is „a bit smaller" than number $y$. Then the belonging of the pair $(x, y)$ to the fuzzy set thus described may be defined, for instance, as follows:

$$\begin{array}{llll} f(1,4)=0{,}5, & f(1,3)=0{,}8, & f(1,2)=1, & f(1,1)=0, \\ f(2,4)=0{,}8, & f(2,3)=1, & f(2,2)=0, & f(2,1)=0, \\ f(3,4)=1, & f(3,3)=0, & f(3,2)=0, & f(3,1)=0, \\ f(4,4)=0, & f(4,3)=0, & f(4,2)=0, & f(4,1)=0. \end{array}$$

Take $x_0=1$. Then

$$f(x_0,x_0)=f(1,1)=0<f(1,4)<f(1,3)<f(1,2).$$

Consequently, $x_0=1$ is not a fixed point of the fuzzy mapping by condition (2). Analogously,

$$f(2,2)=0=f(2,1)<f(2,4)<f(2,3)$$

and

$$f(3,3)=0=f(3,2)=f(3,1)<f(3,4).$$

Whereas $x_0=4$ is a fixed point because

$$f(4,4)=0=f(4,1)=f(4,2)=f(4,3).$$

## Reference

[1] D. Butnariu, Fixed points for fuzzy mappings, Fuzzy Sets and Systems 7 (1982), 191-207.