ENTROPY AND CORRELATION OF PROBABILISTIC FUZZY SETS

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Let $(\Omega, \mathscr{B}, P)$ be a probabilistic space, while $([0, 1], \mathscr{B}_{[0,1]})$ the so-called characteristic space, where $\mathscr{B}_{[0,1]}$ is the Borel field of subsets of the interval $[0, 1]$. Let $X=\{x_1, x_2, ..., x_n\}$ be a set of some objects.

Following K. Hirota [4], we introduce the definition below.

DEFINITION. By a probabilistic set A in X we mean the set described by the determining function

$$f=\mu_A: X\times\Omega\to[0, 1]$$

where $\mu(x, \cdot)$ is a $(\mathscr{B}, \mathscr{B}_{[0,1]})$ – measurable function for each $x \in X$.

1. Entropy and informational energy in the class of probabilistic sets

Let $\Omega=\{\omega_1, \omega_2, ..., \omega_M\}$, $X=\{x_1, x_2, ..., x_N\}$, and let f be a probabilistic set in $X\times\Omega$.

For a fixed $\omega_j \in \Omega$, the probabilistic set f becomes an ordinary fuzzy set. Following A. de Luca and S. Termini [1], the entropy of such a set is expressed by the formula

$$d_j(f)=H(f)+H(f')=K\sum_{i=1}^{N} S(f(x_i, \omega_j))$$

where $H(f)$ is the functional

$$H(f)=-K\sum_{i=1}^{N} f(x_i, \omega_j)\ln f(x_i, \omega_j), \qquad K=\text{const}>0,$$

$f'(x, \omega)=1-f(x, \omega)$ is the complement of the set f, and S is the so-called Shannon function

$$S(x)=\begin{cases}-x\ln x-(1-x)\ln(1-x) & \text{when } x\in(0, 1),\\ 0 & \text{when } x\in\{0, 1\}.\end{cases}$$

The entropy $d(f)$ is some fuzziness measure of the fuzzy set f and may be regarded, analogously to the Shannon entropy, as a quantity measuring the amount of information.

Then the formula

$$D(f)=\sum_{j=1}^{M} P(\omega_j)\cdot d_j(f)=K\sum_{j=1}^{M}\sum_{i=1}^{N} P(\omega_j)\cdot S(f(x_i,\omega_j))$$

expresses the entropy of the probabilistic set f (it is a random variable because it is a function of the random variable $f(x,\omega)$).

In 1977 D. Dumitrescu [2], basing himself on O. Onicescu's notion of informational energy [5], introduced the notion of the informational energy $t(f)$ of the fuzzy set f, defined by the formula

$$t(f)=\sum_{i=1}^{N}[f^2(x_i)+(f'(x_i))^2]=\sum_{i=1}^{N} e(f(x_i))$$

where $e(x)=2x^2-2x+1$.

The above function is used as a sharpness measure of the set f.

With a fixed $\omega_j\in\Omega$, the informational energy of the set f under our consideration is expressed by the formula

$$t_j(f)=\sum_{i=1}^{N} e(f(x_i,\omega_j)).$$

In turn, the formula

$$T(f)=\sum_{j=1}^{M} P(\omega_j)t_j(f)=\sum_{j=1}^{M}\sum_{i=1}^{N} P(\omega_j)\cdot e(f(x_i,\omega_j))$$

expresses a mean informational energy of the probabilistic set f. (It is also a random variable).

It is self-evident that, in the deterministic case, the formula is reduced to $t(f)$.

2. Correlation of probabilistic sets

Let f and g be probabilistic sets defined in $X\times\Omega$, X and Ω being defined as previously.

For a fixed $\omega_j\in\Omega$, the correlation of the sets f and g is defined by the formula

$$c_j(f,g)=\sum_{i=1}^{N}[f(x_i,\omega_j)\cdot g(x_i,\omega_j)+f'(x_i,\omega_j)\cdot g'(x_i,\omega_j)],$$

which is consistent with the idea of D. Dumitrescu in [3], whereas by

$$C(f,g)=\sum_{j=1}^{M} P(\omega_j)\cdot c_j(f,g)$$

we understand a mean correlation of the probabilistic sets f and g.

By the formula

$$K(f,g)=\sum_{j=1}^{M} P(\omega_j)\cdot k_j(f,g),$$

where

$$k_j(f,g)=\frac{c_j(f,g)}{\sqrt{t_j(f)\cdot t_j(g)}}$$

is a correlation coefficient of the fuzzy sets f and g with a fixed ω_j, we define a (mean) correlation coefficient of the probabilistic sets f and g.

PROPOSITION 1. The correlation of the probabilistic sets f and g satisfies the condition

$$C(f,f)=T(f).$$

This follows from the formulae given above.

PROPOSITION 2. It is evident that

$$K(f,f)=1.$$

We have the obvious

PROPOSITION 3. $K(f,g)=K(g,f)$.

PROPOSITION 4. $0\leqq K(f,g)\leqq 1$ since $0\leqq k_j(f,g)\leqq 1$ ([3]).

We have

THEOREM. $C(f,g)=0\Leftrightarrow f(x_i,\omega_j)=0$ and $g(x_i,\omega_j)=1$ for $(x_i,\omega_j)\in S_1$, and $f(x_k,\omega_l)=1$ and $g(x_k,\omega_l)=0$ for $(x_k,\omega_l)\in S_2$, where $S_1\cap S_2=\emptyset$ and $S_1\cup S_2=X\times\Omega$.

The proof follows from the analogous proposition for fuzzy sets (see [3]).

If $C(f,g)=0$, then the probabilistic sets f and g are called uncorrelated.

In the case of three probabilistic sets $f,g,h: X\times\Omega\to[0,1]$ where X and Ω are as before, the correlation of these sets is expressed by the formula

$$C(f,g,h)=\sum_{j=1}^{M} P(\omega_j)\cdot c_j(f,g,h)$$

where

$$c_j(f,g,h)=\sum_{i=1}^{N}[f(x_i,\omega_j)\cdot g(x_i,\omega_j)\cdot h(x_i,\omega_j)+f'(x_i,\omega_j)\cdot g'(x_i,\omega_j)\cdot h'(x_i,\omega_j)]$$

for a fixed $\omega_j\in\Omega$.

The correlation coefficient will then be defined as

$$K(f,g,h)=\sum_{j=1}^{M} P(\omega_j)\cdot k_j(f,g,h)$$

where

$$k_j(f,g,h)=\frac{c_j(f,g,h)}{[t_j(f)\cdot t_j(g)\cdot t_j(h)]^{1/3}}$$

is the correlation coefficient of the sets f,g,h for a fixed $\omega_j\in\Omega$.

The generalization to n probabilistic sets is obvious.

As before, let card $X=N$, card $\Omega=M$ and let f and g be probabilistic sets on $X\times\Omega$. For these sets, we introduce the so-called retarded correlation, at first – with a fixed $\omega_j\in\Omega$, by the formula

$$c_j^{\tau,\varkappa}(f,g)=\sum_{i=1}^{N}[f(x_i,\omega_j)\cdot g(x_{i+\tau},\omega_{j+\varkappa})+f'(x_i,\omega_j)\cdot g'(x_{i+\tau},\omega_{j+\varkappa})]$$

where $\tau\in\{0,1,\ldots,N\}$, $\varkappa\in\{0,1,\ldots,M\}$ and $g(x_{i+\tau},\omega_{j+\varkappa})=0$ for $i+\tau>N$ or $j+\varkappa>M$, and next – with a varying j, by

$$C^{\tau,\varkappa}(f,g)=\sum_{j=1}^{M}P(\omega_j)\cdot c_j^{\tau,\varkappa}(f,g).$$

The retarded correlation coefficient, with a fixed ω_j, is given by

$$k_j^{\tau,\varkappa}(f,g)=\frac{c_j^{\tau,\varkappa}(f,g)}{t_j(f)\cdot t_j(g)},$$

and the formula

$$K^{\tau,\varkappa}(f,g)=\sum_{j=1}^{M}P(\omega_j)\cdot k_j^{\tau,\varkappa}(f,g)$$

defines the retarded correlation coefficient of the sets f and g.

For $\tau=0$ and $\varkappa=0$, we have, of course,

$$C^{0,0}(f,f)=T(f).$$

References

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